The Spectrum of the MFIE and Calderon Preconditioned EFIE for Scattering by Two-Dimensional Non-Smooth Objects

K. Cools* I. Bogaert† F.P. Andriulli‡

Abstract — **The mixed MFIE and Calderón preconditioned EFIE both can be used to accurately model the scattering of time-harmonic electromagnetic waves by two-dimensional perfect electrical conductors. In the case those conductors are bounded by smooth surfaces, the spectra of the linear systems are clustered around a single non-zero finite value. This configuration is optimal for the iterative solution of these systems by iterative algorithms.**

Regrettably, it has been demonstrated that this optimal configuration is lost when the methods are applied to scattering by non-smooth surfaces. In this case, the spectrum tends to spread out, negatively influencing the number of iterations required for iterative solvers to converge.

In this contribution, this spreading out of the spectrum is studied quantitatively. It is shown that even though the spectrum spreads out, it remains bounded away from zero and oriented along the negative real axis. It can be concluded that iterative solution remains an option, even for non-smooth geometries. In the case the geometry is so complicated that the spectrum is bounded away from zero by only a very small distance, further preconditioning may be required. Here, a quasi-block diagonal preconditioner is introduced that will compress the spectrum. It is explained how this preconditioner can be applied efficiently as expansion in a Neumann series.

1 INTRODUCTION

Scattering of time harmonic electromagnetic waves by two-dimensional perfect electrical conductors can be modelled by boundary integral equations. These equations can be reduced to systems of linear equations by application of the boundary element method. Due to their size, the resulting systems are solved by an iterative algorithm. The number of iterations required (and thus the solution time) depends on the spectral properties of the linear systems.

The most prominent and fundamental boundary integral equations are the EFIE and the MFIE. The EFIE is more versatile than the MFIE: (i) it can model scattering by almost perfect conductors by inclusion of a local surface impedance, and (ii) it can model scattering by very thin objects by considering them as open surfaces.

The EFIE and MFIE can be discretised by substituting for the unknwon current an approximation by an expansion in divergence conforming basis functions and subsequently testing the equation with an equal number of curl conforming testing functions. The symmetry of the EFIE allows for the same functions to be used as expansion and testing functions. The inherent skweness of the MFIE, on the other hand, calls out for testing functions that are dual to the expansion functions.

When applied to smooth scatterers the spectra of linear systems resulting upon application of the boundary element method are well-known. The spectrum of the matrix resulting upon discretization of the EFIE operator T comprises two branches: one branch is approximately located along the positive imaginary axis and extents from a finite value to an accumulation point at infinity, the other branch is located along the negative imaginary axis and extents from a finite value to an accumulation point at zero. When the discretization density is increased, the spectrum will contain eigenvalues that approximate more closely the accumulation points at infinity and zero, respectively. This constellation of eigenvalues is detrimental for the efficient solution of the linear system by iterative algorithms. The boundary element approximation of the EFIE is said to suffer from dense grid breakdown. The spectrum of the matrix resulting upon discretization of the MFIE operator $K - 1/2$ comprises only one branch that accumulated at $-1/2$. This is a consequence of the fact that K is a compact operator, whose spectrum accumulates at 0. Increasing the discretization density will result in the resolution of eigenvalues that are increasingly closer to $-1/2$. This will not significantly affect the number of iterations required for iterative algorithms to arrive at an approximate solution of the linear system; the MFIE does not suffer from dense grid breakdown.

*Electrical Systems and Optics Division, University of Nottingham, NG7 2RD Nottingham, United Kingdom, e-mail: kristof.cools@nottingham.ac.uk, tel.: +44 115 95 15537

†Dept. of Information Technology, Ghent University, Sint-Pietersnieuwstraat, 9000 Gent, Belgium, e-mail: ignace.bogaert@intec.ugent.be

‡Microwave Dept., TELECOM Bretagne, 655 Avenue du Technopole, 29280 Plouzan, France

The spectrum of the EFIE can be regularized by leveraging the self-regularizing properties of the single layer potential of electromagnetics, summarized by the first Calderon identity

$$T^2 = K^2 - \frac{1}{4} = \left(K + \frac{1}{2}\right)\left(K - \frac{1}{2}\right). \quad (1)$$

This identity suggest that the spectrum of T^2 has a single branch accumulating at $-1/4$, and is not subject to dense grid breakdown. Discretization of this identity by the boundary element method involves the discretization of T using both a primal and dual set of functions and the inversion of the Gram matrix between dual and primal functions [1].

When the surface is no longer smooth, not much is known about the spectrum of these operators. Numerical experiments have shown that the spectrum of both the MFIE and the Calderón preconditioned EFIE are more complicated than in the case of smooth surfaces, and that the efficiency of iterative algorithms deteriorates. The spectrum is more complicated because the double layer boundary integral operator K ceases to be compact

In this contribution, it is shown that the spectrum of the double layer boundary integral operator applied to non-smooth scatterers has a continuous part. It is demonstrated that even though the resulting spectrum is not optimal for the iterative solution of the linear system, the spectrum remains concentrated along the negative real axis, centred around the accumulation point $-1/2$, and bounded away from zero. As a result, iterative solution remains a viable strategy. It is shown that the distance at which the spectrum is bounded away from zero depends on the most acute angle in the scatterers surface. For those case where this angle is so small that iterative solution becomes prohibitively expensive, a preconditioning strategy is suggested. Throughout the paper, a $e^{i\omega t}$ time dependence is assumed and suppressed.

2 BOUNDARY INTEGRAL EQUATIONS

Consider a perfectly electrically conducting two-dimensional scatterer with boundary Γ. The boundary Γ can be parametrised by the arc length s with corresponding tangential unit vector $\hat{\boldsymbol{s}}$. The exterior normal unit vector on Γ is denoted $\hat{\boldsymbol{n}}$. The scatterer is submerged in a background medium characterized by a permittivity ϵ and a permeability μ. Let $k = \omega\sqrt{\epsilon\mu}$ and $\eta = \sqrt{\frac{\mu}{\epsilon}}$ be the corresponding wave number and characteristic impedance, respectively. Currents on the contour can be either

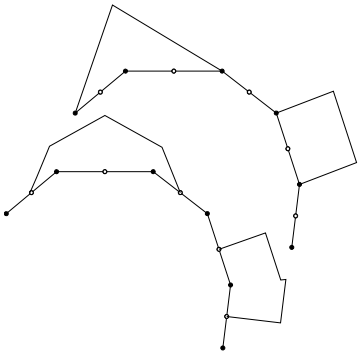

Figure 1: Top: primal continuous, piecewise linear functions and piecewise constant functions. Bottom mesh: dual continuous, piecewise linear and piecewise constant functions.

longitudinal ($\boldsymbol{j} = j_z\hat{\boldsymbol{z}}$) or transversal ($\boldsymbol{j} = j_s\hat{\boldsymbol{s}}$). The electric and magnetic fields radiated by these currents at a point just inside the scatterer is computed as follows:

$$\frac{1}{\eta}e_z(\boldsymbol{r}) = T_k^{zz}(j_z)(\boldsymbol{r}) = -ik\int_\Gamma g(k|\boldsymbol{r}-\boldsymbol{r}'|)j_z\boldsymbol{r}', \quad (2)$$

$$\frac{1}{\eta}e_s(\boldsymbol{r}) = T_k^{ss}(j_s)(\boldsymbol{r}) = -ik\int_\Gamma g(k|\boldsymbol{r}-\boldsymbol{r}'|)j_s\boldsymbol{r}' + \frac{1}{ik}\partial_s\int_\Gamma g(k|\boldsymbol{r}-\boldsymbol{r}'|)\partial_{s'}j_s\boldsymbol{r}', \quad (3)$$

$$h_s(\boldsymbol{r}) = \left(K_k^{sz} - \frac{1}{2}\right)(j_z)(\boldsymbol{r}) = -\int_\Gamma \partial_n g(k|\boldsymbol{r}-\boldsymbol{r}'|)j_z(\boldsymbol{r}') - \frac{1}{2}j_z(\boldsymbol{r}), \quad (4)$$

$$h_z(\boldsymbol{r}) = \left(K_k^{zs} - \frac{1}{2}\right)(j_s)(\boldsymbol{r}) = -\int_\Gamma \partial_{n'} g(k|\boldsymbol{r}-\boldsymbol{r}'|)j_s(\boldsymbol{r}') + \frac{1}{2}j_s(\boldsymbol{r}), \quad (5)$$

where $g(k|\boldsymbol{r}-\boldsymbol{r}'|) = -\frac{i}{4}H_0^{(2)}(k|\boldsymbol{r}-\boldsymbol{r}'|)$, with $H_0^{(2)}$ the Hankel functions of order 0 and second kind. The boundary integral equations describing exterior scattering are then:

$$T_k^{zz}j_z = -\frac{1}{\eta}e_z^i, \quad (6)$$

$$T_k^{ss}j_s = -\frac{1}{\eta}e_z^i, \quad (7)$$

$$\left(K_k^{sz} - \frac{1}{2}\right)j_z = -h_s^i \quad (8)$$

$$\left(K_k^{zs} + \frac{1}{2}\right)j_s = -h_z^i. \quad (9)$$

The discretisation of these equations is done in three steps and according to the considerations in

[2]. First, a mesh $\mathcal{T}$ of segments is defined on Γ. The dual $\mathcal{T}'$ to this mesh is realized through the barycentric refinement τ of $\mathcal{T}$. Next, the tangential current is approximated by a continuous piecewise linear function and the longitudinal current is approximated by a piecewise constant function. Finally, the tangential fields are tested with piecewise constant functions defined on $\mathcal{T}'$ and longitudinal fields are tested with continuous, piecewise linear functions defined on $\mathcal{T}'$. The classic vertex and segment based bases of these spaces are depicted in Fig. 1. The resulting linear systems are

$$\mathsf{T}_k^{zz}\mathsf{j}_z = -\frac{1}{\eta}\mathsf{e}_z^i, \tag{10}$$

$$\mathsf{T}_k^{ss}\mathsf{j}_s = -\frac{1}{\eta}\mathsf{e}_z^i, \tag{11}$$

$$\left(\mathsf{K}_k^{sz} - \frac{1}{2}\mathsf{G}\right)\mathsf{j}_z = -\mathsf{h}_s^i \tag{12}$$

$$\left(\mathsf{K}_k^{zs} + \frac{1}{2}\mathsf{H}\right)\mathsf{j}_s = -\mathsf{h}_z^i. \tag{13}$$

As mentioned in Section 1 and as detailed in [1] for the three-dimensional case, the discretization schemes leading to (10) and (11) suffer from dense grid breakdown. The schemes can be regularised by application of a Calderón multiplicative preconditioned:

$$\mathsf{G}^{-1}\mathsf{T}^{ss}\mathsf{G}^{-*}\mathsf{T}^{zz}\mathsf{j}_z = -\frac{1}{\eta}\mathsf{G}^{-1}\mathsf{T}^{ss}\mathsf{G}^{-*}\mathsf{e}_z^i, \tag{14}$$

$$\mathsf{H}^{-1}\mathsf{T}^{zz}\mathsf{H}^{-*}\mathsf{T}^{ss}\mathsf{j}_s = -\frac{1}{\eta}\mathsf{H}^{-1}\mathsf{T}^{zz}\mathsf{H}^{-*}\mathsf{e}_z^i. \tag{15}$$

These regularised equations are more complicated than the MFIEs, but they are indispensible for the simulation of scattering by thin conductors, almost perfect conductors described by a surface impedance, and as an ingredient in resonance free, low-frequency/dense grid stable combined field integral equations [3].

3 SPECTRAL PROPERTIES

For smooth surfaces, the eigenvalues of (12) and (13) cluster around $-1/2$. The eigenvalues of (14) and (15) cluster around $-1/4$. This behaviour is ideal for the efficient solution of the linear systems by means of iterative algorithms. It has been reported [4] that for objects with sharp corners, the spectrum spreads out and the efficiency of the Calderón preconditioner deteriorates.

It is assumed that iterative solution is possible as long as all of the eigenvalues of the system, safe maybe a finite number of them (a number bounded uniformly with respect to the mesh density) are concentrated along a single half axis in the complex plane. In two dimensions, it turns out that the deterioration of the spectrum as described in previous paragraph does not render iterative solution impossible.

In fact, in [5], the effect of the presence of corners on the spectrum of K^{ss} and K^{zz}, which are compact in the absence of corners, is discussed in detail. Assume that the sharpest corner (measured with respect to the interior of the scatterer) in the geometry is $\alpha = (1-\chi)\pi$, where χ is a parameter that can be between 0 and 1. The spectrum of K_0^{ss} and K_0^{zz}, applied to only a small neighbourhood of the corner in Γ will be continuous and spanning $\frac{1}{2}[-\chi, \chi]$. As a consequence, the MFIE operator's spectrum will contain a continuous part extending over $-\frac{1}{2} - \frac{1}{2}[-\chi.\chi]$. Since $\chi < 1$ the spectrum will always be bounded away from zero, uniformly with respect to the discretisation density. This implies that away from resonances, the MFIE will be invertible by means of iterative solution methods.

These observations in turn have consequences for the spectrum of the Calderón preconditioned EFIE. Given the first Calder'on identity (that splits in two identities due to the TM/TE subdivision)

$$T^{zz}T^{ss} = \left(K^{zs} + \frac{1}{2}\right)\left(K^{zs} - \frac{1}{2}\right), \tag{16}$$

$$T^{ss}Z^{zz} = \left(K^{sz} + \frac{1}{2}\right)\left(K^{sz} - \frac{1}{2}\right), \tag{17}$$

and the fact that the spectrum of $K^{sz/zs} + 1/2$ is the same as the spectrum of $K^{sz/zs} - 1/2$ shifted to the right by one, the spectrum of the Calder'on preconditioned EFIE too can be seen to contain a continuous part. More in particular, the continuous spectrum extents over the image of $-\frac{1}{2} + \frac{1}{2}(-\chi, \chi)$ under

$$\sigma \to \left(\sigma + \frac{1}{2}\right)\left(\sigma - \frac{1}{2}\right) = -\frac{1}{4} + \sigma^2, \tag{18}$$

which is $-\frac{1}{4} + \frac{1}{4}[0, \chi^2]$. As is the case for the MFIE spectrum, the Calderón preconditioned spectrum is bounded away from zero, uniformly with respect to the mesh density. Moreover, the spectrum does not extend left from $-1/4$.

4 PRECONDITIONER

Even though the spectra of the boundary integral operators remain bounded away from zero, it might be advantageous to apply a preconditioner designed to compress the continuous spectrum. In [5], a preconditioner is suggested for the static MFIE. Here this preconditioner is adapted to the context

of the dynamic MFIE, discretized using a mixed scheme. The regularizing operator is chosen to be $\left(K^{sz/zs}_{k-i\alpha} - 1/2\right)^{-1}$, with α a Yukawa parameter inserted to smoothly truncate the extent of the Green's function's reach. This operator is approximately block diagonal with non-zero blocks only at the corners. This is because on one hand self-interactions of smooth segments through the double layer boundary integral operator are very weak, and on the other hand any non-self interactions are disabled due to the exponential decay of the Yukawa type Green's function. This operator is discretized in the usual manner, giving rise to the preconditioners $\left(\mathbf{K}^{sz}_{k-i\alpha} - \frac{1}{2}\mathbf{G}\right)^{-1}$ and $\left(\mathbf{K}^{zs}_{k-i\alpha} + \frac{1}{2}\mathbf{H}\right)^{-1}$, for (12) and (13) respectively. It is demonstrated in the numerical results section that for this preconditioner the Yukawa parameter α can be tuned such that the matrix 2-norm of $\mathbf{K}^{sz/zs}_{k-i\alpha}$ is smaller than that of $\frac{1}{2}$. As a result, the preconditioner can be approximated by Neumann series:

$$\left(\mathbf{K}^{sz}_{k-i\alpha} - \frac{1}{2}\mathbf{G}\right)^{-1} \approx -2\sum_{k=0}^{n}\left(2\mathbf{G}^{-1}\mathbf{K}^{sz}_{k-i\alpha}\right)^{k}\mathbf{G}^{-1} \tag{19}$$

$$\left(\mathbf{K}^{zs}_{k-i\alpha} + \frac{1}{2}\mathbf{H}\right)^{-1} \approx 2\sum_{k=0}^{n}\left(-2\mathbf{H}^{-1}\mathbf{K}^{zs}_{k-i\alpha}\right)^{k}\mathbf{H}^{-1} \tag{20}$$

5 NUMERICAL RESULTS

Consider the triangle in Fig. 2. The contour is subdivided in 402 segments. The interactions matrices $\mathbf{T}^{zz}_{k}$, $\mathbf{T}^{ss}_{k}$, $\mathbf{K}^{sz}_{k}$, and $\mathbf{K}^{sz}_{k-i\alpha}$ for $k = 2\pi/5$ and $\alpha = -1$ and the Gram matrix $\mathbf{G}$ are computed. The spectra of the TE MFIE operator and the TE CP-EFIE are computed and plotted together with the theoretical expectations (Fig. 3). The theoretical spectra and numerical outcomes agree well. Next, the MFIE is preconditioned using the methods described in the previous paragraph. The Neumann series expansion is valid since $\|\mathbf{G}/2\| = 1$ and $\|K^{sz}_{k-i\alpha}\| = 0.9091$. The preconditioned operator's spectrum for a 1-term and 5-term preconditioner is plotted. The spectrum is increasingly compressed as the number of terms is increased (Fig. 3).

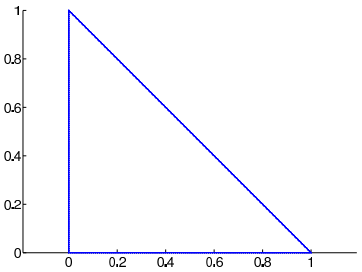

Figure 2: Non-smooth geometry. The sharpest interior corneris $\pi/4$, corresponding to $\chi = 3/4$.

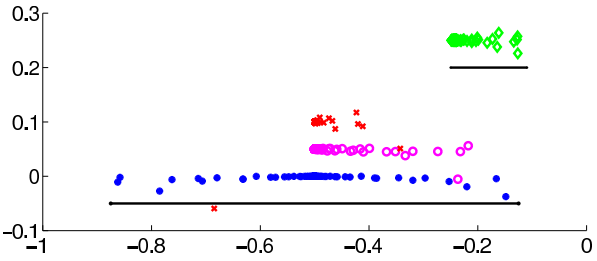

Figure 3: Eigenvalue distribution in the complex plane. The spectra are shifted apart in the imaginary direction for clarity. From bottom to top: spectrum of MFIE, 1-term preconditioned MFIE, 5-term preconditioned MFIE, Calderón preconditioned EFIE. Black lines are theoretical predictions.

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